Reese Ch. 21

(Sects. 4-5 = pp. 965-971)

- 3 Important Theorems Involving "Del"
- Integral Form of Maxwell's Equations
- Differential Form of Maxwell's Equations
- Maxwell's Equations
  away from Charges and Currents
- Propagating Wave Solutions

Important Theorems Involving Del: I

The gradient theorem:

$$f(\vec{r}_2) - f(\vec{r}_1) = \int\limits_{\substack{\text{any path}\\(1\to 2)}} \left[\vec{\nabla} f(\vec{r})\right] \cdot d\vec{r}$$

Important Theorems Involving Del: II

The divergence theorem:

$$\underset{\substack{\text{closed}\\\text{surface}}}{\oint\!\!\oint} \vec{u}(\vec{r}) \cdot d\vec{S} = \underset{\substack{\text{enclosed}\\\text{volume}}}{\oint\!\!\oint\!\!\oint} \left[\vec{\nabla} \cdot \vec{u}(\vec{r})\right] dV$$

Important Theorems Involving Del: III

Stokes' theorem (= the curl theorem):

$$\underset{\substack{\text{closed}\\\text{curve}}}{\oint} \vec{u}(\vec{r}) \cdot d\vec{r} = \underset{\substack{\text{any bounded}\\\text{surface (open)}}}{\iint} \left[\vec{\nabla} \times \vec{u}(\vec{r})\right] \cdot d\vec{S}$$

Integral Form of Maxwell's Equations: I

Gauss's Law for the electric field:

$$\underset{\text{closed surface}}{\oiint} \vec{E}(\vec{r}) \cdot d\vec{S} = \frac{Q_{\text{net enclosed}}}{\varepsilon_o} = \frac{1}{\varepsilon_o} \underset{\text{enclosed volume}}{\iiint} \rho(\vec{r})\,dV$$

Integral Form of Maxwell's Equations: II

Gauss's Law for the magnetic field:

$$\underset{\text{closed surface}}{\oiint} \vec{B}(\vec{r}) \cdot d\vec{S} = 0$$

Integral Form of Maxwell's Equations: III

The Ampere-Maxwell Law:

$$\underset{\text{closed curve}}{\oint} \vec{B}(\vec{r}) \cdot d\vec{r} = \mu_o (I + I_D) \Rightarrow$$

$$\underset{\text{closed curve}}{\oint} \vec{B}(\vec{r}) \cdot d\vec{r} = \mu_o \left[\underset{\text{any bounded surface (open)}}{\iint} \vec{J} \cdot d\vec{S} \right] + \mu_o \varepsilon_o \frac{d}{dt} \left[\underset{\text{SAME bounded surface (open)}}{\iint} \vec{E} \cdot d\vec{S} \right]$$

Integral Form of Maxwell's Equations: IV

Faraday's Law of Induction:

$$\underset{\text{closed curve}}{\oint} \vec{E}(\vec{r}) \cdot d\vec{r} = -\frac{d}{dt} \left[\underset{\text{any bounded surface (open)}}{\iint} \vec{B} \cdot d\vec{S} \right]$$

Differential Form of Maxwell's Equations: I

Gauss's Law for the electric field:

$$\oiint_{\substack{\text{closed}\\\text{surface}}} \vec{E}(\vec{r})\cdot d\vec{S} = \frac{Q_{\text{net enclosed}}}{\varepsilon_o} = \frac{1}{\varepsilon_o}\iiint_{\substack{\text{enclosed}\\\text{volume}}} \rho(\vec{r})dV \text{ combined with}$$

$$\oiint_{\substack{\text{closed}\\\text{surface}}} \vec{u}(\vec{r})\cdot d\vec{S} = \iiint_{\substack{\text{enclosed}\\\text{volume}}} \left[\vec{\nabla}\cdot\vec{u}(\vec{r})\right]dV \quad \Rightarrow$$

$$\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t) = \frac{\rho(\vec{r},t)}{\varepsilon_o}}$$

Differential Form of Maxwell's Equations: II

Gauss's Law for the magnetic field:

$$\oiint_{\substack{\text{closed}\\\text{surface}}} \vec{B}(\vec{r})\cdot d\vec{S} = 0 \text{ combined with}$$

$$\oiint_{\substack{\text{closed}\\\text{surface}}} \vec{u}(\vec{r})\cdot d\vec{S} = \iiint_{\substack{\text{enclosed}\\\text{volume}}} \left[\vec{\nabla}\cdot\vec{u}(\vec{r})\right]dV \quad \Rightarrow$$

$$\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0}$$

Differential Form of Maxwell's Equations: III

The Ampere-Maxwell Law:

$$\oint_{\substack{\text{closed}\\\text{curve}}} \vec{B}(\vec{r})\cdot d\vec{r} = \mu_o\left[\iint_{\substack{\text{any bounded}\\\text{surface (open)}}} \vec{J}\cdot d\vec{S}\right] + \mu_o\varepsilon_o\frac{d}{dt}\left[\iint_{\substack{\text{SAME bounded}\\\text{surface (open)}}} \vec{E}\cdot d\vec{S}\right] \text{ combined with}$$

$$\oint_{\substack{\text{closed}\\\text{curve}}} \vec{u}(\vec{r})\cdot d\vec{r} = \iint_{\substack{\text{any bounded}\\\text{surface (open)}}} \left[\vec{\nabla}\times\vec{u}(\vec{r})\right]\cdot d\vec{S} \Rightarrow$$

$$\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu_o\vec{J}(\vec{r},t) + \mu_o\varepsilon_o\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$$

Differential Form of Maxwell's Equations: IV

Faraday's Law of Induction:

$$\oint_{\substack{\text{closed}\\\text{curve}}} \vec{E}(\vec{r})\cdot d\vec{r} = -\frac{d}{dt}\left[\iint_{\substack{\text{any bounded}\\\text{surface (open)}}} \vec{B}\cdot d\vec{S}\right] \text{ combined with}$$

$$\oint_{\substack{\text{closed}\\\text{curve}}} \vec{u}(\vec{r})\cdot d\vec{r} = \iint_{\substack{\text{any bounded}\\\text{surface (open)}}} \left[\vec{\nabla}\times\vec{u}(\vec{r})\right]\cdot d\vec{S} \Rightarrow$$

$$\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$$

Maxwell's Equations *in vacuum* (away from Charges and Currents)

I. Gauss E $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t)=0}$

II: Gauss B $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0}$

III: Ampere-Maxwell $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu_o\varepsilon_o\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

IV: Faraday $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$

Propagating Wave Solutions: I

Notice that the permittivity ε_o and the permeability μ_o appear in the third equation as the product $\mu_o\varepsilon_o$. What is its SI value?

$$\mu_o\varepsilon_o=(4\pi\times10^{-7}\frac{T\cdot m}{A})\times(8.854\times10^{-12}\frac{C^2}{N\cdot m^2})$$

$$\Rightarrow \mu_o\varepsilon_o=0.1113\times10^{-16}\frac{s^2}{m^2}\equiv\frac{1}{c^2}$$

where the constant $\boxed{c\equiv\left(\mu_o\varepsilon_o\right)^{-1/2}=2.998\times10^8\frac{m}{s}}$

Propagating Wave Solutions: II

I. Gauss E $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t)=0}$

II: Gauss B $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0}$

III: Ampere-Maxwell $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\frac{1}{c^2}\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

IV: Faraday $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$

Propagating Wave Solutions: III

- These equations allow many solutions – let's look for the simplest of them.
  - Assume that the fields only depend on ONE spatial coordinate, which we choose to be x.
  - Assume that the electric field has zero x and z components: $\boxed{\vec{E}(\vec{r},t)=\hat{j}E_y(x,t)}$
  - Assume that the magnetic field has zero x and y components: $\boxed{\vec{B}(\vec{r},t)=\hat{k}B_z(x,t)}$

Propagating Wave Solutions: IV

I. Gauss E $\Rightarrow 0 = 0$

II: Gauss B $\Rightarrow 0 = 0$

III: Ampere-Maxwell $\boxed{\Rightarrow -\frac{\partial B_z}{\partial x} = \frac{1}{c^2}\frac{\partial E_y}{\partial t}}$

IV: Faraday $\boxed{\Rightarrow \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}}$

Propagating Wave Solutions: V

- Differentiate III with respect to t and IV with respect to x:

III: $\boxed{\Rightarrow -\frac{\partial^2 B_z}{\partial t \partial x} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}}$

IV: $\boxed{\Rightarrow \frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial x \partial t}}$

Propagating Wave Solutions: VI

- Mathematical theorem: Second-order mixed-partial derivatives are equal, so . . .
- BINGO! $\boxed{\Rightarrow \frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}}$
- This is "the wave equation", for which the solution is $\boxed{E_y(x,t) = f(x-ct) + g(x+ct)}$

  where f(u) & g(u) are <u>arbitrary</u> functions.

Propagating Wave Solutions: VII

- If we choose g=0, then we get a rightward moving wave:

$$\boxed{\begin{array}{l}\vec{E}(x,t) = \hat{j}E_y(x,t) = \hat{j}f(x-ct) \text{ and} \\ \vec{B}(x,t) = \hat{k}B_z(x,t) = +\hat{k}c^{-1}f(x-ct)\end{array}}$$

- If we choose f=0, then we get a leftward moving wave:

$$\boxed{\begin{array}{l}\vec{E}(x,t) = \hat{j}E_y(x,t) = \hat{j}g(x+ct) \text{ and} \\ \vec{B}(x,t) = \hat{k}B_z(x,t) = -\hat{k}c^{-1}g(x+ct)\end{array}}$$

- NOTE: **E x B** gives the direction of propagation