Reese Ch. 9, Sect. 3 (pp. 388-390)

- The Rocket: A Variable-Mass System
  - Terminology
  - Analysis of Single-Stage Rocket
- Multi-Stage Rockets

Rocket Terminology

- The **exhaust speed** is the speed of the exiting exhaust gases *relative to the nozzle*. In our analysis, we'll assume that this speed is constant throughout the operation of any given engine.
- A **single-stage rocket** has only one engine; a **multi-stage rocket** has two or more. (Why use a multi-stage rocket?)
- The **payload mass** (often called just the payload) is the mass of that part of the rocket that can be used for purposes other than propulsion. (This is not same as the final mass of the rocket. Why not?)

Analysis of Single-Stage Rocket:I

- For simplicity, assume straight-line motion in gravity-free environment, with m(t) and $v_x(t)$ being the instantaneous mass and velocity of the rocket. Newton's Laws (or conservation of momentum applied to the entire system) yield the following equation for the rocket's motion:

$$\boxed{m(t)\frac{dv_x(t)}{dt} = -v_{exh}\frac{dm(t)}{dt}}$$

Analysis of Single-Stage Rocket:II

- Integrating this equation from the beginning of the burn until the end of the burn, we obtain

$$\boxed{v_{xf} - v_{xi} = v_{exh}\ln\left(\frac{m_i}{m_f}\right)}$$

- Because the exhaust speeds of chemical rockets are limited to a few km/s while the speed for escape from Earth is 11.2 km/s, this equation reveals the need for multistage rockets for space missions.

Multi-Stage Rockets

- For a multi-stage rocket, the rocket equation gives the velocity "boost" from each stage. The velocity at the beginning of stage n+1 is just the velocity at the end of stage n, but the mass at the beginning of stage n+1 is LESS than the mass at the end of stage n. (How can this be? It's because the tanks, engine, and nozzle of stage n are jettisoned.)
- Final result:

$$\boxed{v_{xf} - v_{xi} = \sum_{n=1}^{N} v_{exh,n} \ln\left(\frac{m_{i,n}}{m_{f,n}}\right)}$$

Reese Ch. 10, part A1
(pp. 425-436)

- Orbital Motion vs. Spin
- Orbital Angular Momentum
- Circular Orbital Motion
- Noncircular Orbital Motion

Spin vs. Orbital Motion

- **Orbital motion** is motion of an extended object (or collection of objects) "as a whole" – the translational motion of the object's center-of-mass point.
- **Spin** is rotational motion of an object – the turning of an object around its center of mass point.
- Sometimes the distinction is fuzzy – what we call a given motion can depend on how much we mentally "lump" small objects together into larger wholes.

Orbital Angular Momentum of a Particle

- The orbital angular momentum of a particle is defined as $\vec{L} \equiv \vec{r} \times \vec{p} = m\left(\vec{r} \times \vec{v}\right)$
- Notes:
  - The orbital angular momentum depends on the particle's mass, speed, distance from the origin, and the angle between its velocity vector and its position vector.
  - If the particle is moving in the xy plane and CCW about the origin, then the angular momentum vector points in the +z direction.

What causes a particle's angular momentum to change?

- Suppose that a single force acts on a particle. It is not difficult to show that

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \equiv \vec{\tau}$$

- Notes:
  - $\vec{\tau}$ is called the torque acting on the particle
  - If several forces act on the particle, the correct value to use for the effective torque is just the vector sum of the torques exerted by the various forces: $\vec{\tau}_{net}$

Alternate Expressions for the Torque Exerted by a Force: I

- The line through the object along the direction of any force vector is called the line of action of that force.
- The (perpendicular) distance from the origin to the line of action is called the moment arm of the force.
- The magnitude of the torque equals the moment arm times the magnitude of the force:

$$\left|\vec{\tau}\right| = r_{\perp}\left|\vec{F}\right|$$

Alternate Expressions for the Torque Exerted by a Force: II

- The magnitude of the torque can also be written as the full distance from the origin to the particle, times the tangential component of the force (note that this component is *perpendicular* to the radius vector): $\left|\vec{\tau}\right| = rF_{\perp}$
- The magnitude of the torque can also be written using the angle between the two vectors:

$$\tau = rF\left|\sin\theta\right|$$

Uniform Circular Motion (UCM)

- This name refers to "motion in a circular path at constant speed."
- This motion is ***periodic.*** What is the period T?
  - If the radius of the path is r, the circumference is $2\pi r$.
  - If the speed is v, then $vT = 2\pi r$, so . . .

$$T = 2\pi r/v$$

r and θ versus time for UCM

- Assuming that. . .
  - path lies in xy-plane
  - coordinate origin is at the center of the circle
  - when t = 0 the particle has $x_i = r$ and $v_{yi} = v\ (>0)$
- Then the particle's polar coordinates are:

  r (a constant!) ***and*** $\theta(t) = (v/r)t = 2\pi ft = \omega t$, where

  $f = v/2\pi r$ is called the ***(ordinary) frequency***

  $\omega = 2\pi f$ is called the ***angular frequency***

Radial and tangential unit vectors

- In treating motions which exhibit circular symmetry (such as UCM), it can be really convenient to use two new unit vectors:
  - the unit vector $\mathbf{e}_r$ which points radially outward
  - the unit vector $\mathbf{e}_\theta$ which is 90º counterclockwise (CCW) from $\mathbf{e}_r$ so that it points tangentially in direction of increasing θ

Achtung!

- The directions of these unit vectors depend on location in the xy-plane.
- Example #1: At location $\mathbf{r} = 2\mathbf{i} + 0\mathbf{j}$ . . .

  $\mathbf{e}_r = \mathbf{i}$ and $\mathbf{e}_\theta = \mathbf{j}$
- Example #2: At location $\mathbf{r} = 5\mathbf{i} + 5\mathbf{j}$ . . .

  $\mathbf{e}_r = (\mathbf{i} + \mathbf{j})/\sqrt{2}$ and $\mathbf{e}_\theta = (-\mathbf{i} + \mathbf{j})/\sqrt{2}$
- Example #3: At location $\mathbf{r} = 0\mathbf{i} - 4\mathbf{j}$ . . .

  $\mathbf{e}_r = -\mathbf{j}$ and $\mathbf{e}_\theta = \mathbf{i}$

r, **v**, and **a** vectors in UCM

- radius: $\mathbf{r}$ = r times $\mathbf{e}_r$
- velocity vector: $\mathbf{v} = v_\theta$ times $\mathbf{e}_\theta = v_\theta\, \mathbf{e}_\theta$
  - speed $v = |v_\theta|$
  - $\mathbf{v} = v\, \mathbf{e}_\theta$ *(for CCW motion)*
  - $\mathbf{v} = -v\, \mathbf{e}_\theta$ *(for CW motion)*
- acceleration vector: $\mathbf{a} = -(v^2/r)\mathbf{e}_r$

Non-uniform circular motion (NUCM)

- Radius: still have **r** = r times $\mathbf{e}_r$
- The velocity vector is still purely tangential:
  - **v** = $v_\theta(t)$ times $\mathbf{e}_\theta = v_\theta(t)\mathbf{e}_\theta$
- The speed is a function of time: $v(t) = | v_\theta(t) |$
- Now the acceleration has two components:
- **a** = a_r times $\mathbf{e}_r$ + a_θ times $\mathbf{e}_\theta = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta$
  - the radial component: $a_r(t) = - v^2/r$
  - the tangential component: $a_\theta (t) = dv_\theta/dt$

Dynamics of Circular Motion

- We simply apply Newton's Second Law: in order to move with a particular acceleration **a**, an object of mass m ***must*** be moving under the influence of force(s) whose resultant is m times **a**:

$$\mathbf{F}_{net} = m\mathbf{a}$$

Uniform Circular Motion (UCM)

- At each instant, the acceleration vector points radially inward. It has a constant magnitude v^2/r.
- Therefore the instantaneous net force **must** point radially inward and have a magnitude mv^2/r. This is usually called the "centripetal force."
- Beware: Centripetal force is just a name for the necessary resultant of the physical forces that are acting. It does **NOT** belong in a free-body diagram.

Angular Momentum for UCM

- Using the center of the circle as the coordinate origin, what is the angular momentum vector for a particle exhibiting counterclockwise UCM in the xy plane?
- How about for clockwise UCM?
- In either case, does the angular momentum vary with time?
- Is this consistent with rotational version of Newton's 2nd Law?
- How does this connect with what you've already learned about circular motion under gravity?

Nonuniform Circular Motion (NUCM)

- The instantaneous acceleration has a radial component $-[v(t)]^2/r$ and a tangential component dv_θ/dt.
- Therefore the instantaneous net force **must** have components that are just m times these acceleration components.
- Why can't a satellite exhibit NUCM?

Noncircular Orbital Motion under Gravity

- When an object follows a noncircular orbit under the influence of gravity, what does the rotational version of Newton's 2^{nd} Law tell us about the object's angular momentum?
- How is the rotational form of Newton's 2^{nd} Law related to Kepler's 2^{nd} Law (the "equal areas" law)?

Reese Ch. 10, part A2 (pp. 436-445)

- Rigid Bodies
- Spin Angular Momentum
- Rotational Inertia of Symmetric Bodies
- Equation for Rate of Change of Spin
- Kinetic Energy due to Spin
- Rotational Distortion of Objects

Rigid Bodies

- A rigid body is an object which exhibits no deformation (change of shape). In a rigid body, the distance between any two bits of material (any two atoms, say) remains constant.
- Most of our study of rotational dynamics will deal with rigid bodies that have at least one axis of symmetry.

Spin Angular Momentum ("Spin") of a Rigid Body

- For simplicity, consider a rigid body whose center of mass (CM) is at rest and whose rotation axis coincides with one of body's axes of symmetry. Then
  - Each of the atoms in the body is in uniform circular motion about the rotation axis, and all of the atoms have the same angular velocity.
  - If we use the CM as the coordinate origin and add up the (orbital) angular momentum vectors for all of the atoms, we get a simple result (after doing vector algebra and utilizing the system's symmetry):

$$\vec{L}_{spin} = \left(\sum_i m_i r_{i\perp}^2 \right) \vec{\omega}$$

Comments on Spin

- The summation in the spin equation is called the <u>rotational inertia</u> (or "moment of inertia") of the object:

$$I \equiv \left(\sum_i m_i r_{i\perp}^2 \right)$$

- The rotational inertia I
  - depends on the total mass of the object and how the mass is distributed, and
  - also depends on the specific symmetry axis about which the body is rotating.

Evaluating Rotational Inertia:I

- For a body consisting of several individual mass points connected by "massless" rods, we can directly apply the discrete sum:

$$I \equiv \left(\sum_i m_i r_{i\perp}^2 \right)$$

- For a body in which the matter is distributed continuously, we must interpret the sum as an integral:

$$I \equiv \int_{body} r_\perp^2 dm = \int_{body} r_\perp^2 \rho(\vec{r}) dV$$

Evaluating Rotational Inertia:II

- For rigid assemblies whose parts are simple rigid bodies, we can simply add up the I-values for the parts (provided that the rotation axis coincides with a symmetry axis for each part):

$$I_{assembly} = \sum_i I_i$$

- Table 10.1 (p. 440) lists the rotational inertia values for some common symmetric bodies.

Rotational Form of Newton's 2nd Law for a Spinning System: I

- We previously obtained the equation $\frac{d\vec{L}}{dt} = \vec{\tau}_{net}$ for a single orbiting particle, and we have also defined the spin angular momentum of a rotating body as the vector sum of the orbital angular momenta of its various parts.
- These two facts mean that if we let $\vec{\tau}_{i(net)}$ denote the net torque on particle *i* and we define the total torque on the system as $\vec{\tau}_{total} \equiv \sum_i \vec{\tau}_{i(net)}$, then we get the rotational 2nd law for a spinning system: $$\frac{d\vec{L}_{spin}}{dt} = \vec{\tau}_{total}$$

Rotational Form of Newton's 2nd Law for a Spinning System: II

- The net torque $\vec{\tau}_{i(net)}$ on any one particle consists of an internal contribution (due to forces from other particles in the object) and an external contribution. In many situations the internal forces act along the line between the interacting particles, which means the total internal torque is zero. We only consider such cases, so we can write

$$\frac{d\vec{L}_{spin}}{dt} = \vec{\tau}_{total(external)}$$

Rotational Form of Newton's 2nd Law for a Spinning System: III

- Combining this equation with the expression for the spin angular momentum as the product of rotational inertia and angular velocity, we obtain

$$\frac{d\vec{L}_{spin}}{dt} = \frac{d(I\vec{\omega})}{dt} = \vec{\tau}_{total(external)}$$

Kinetic Energy due to Spin

- To obtain the total kinetic energy of a spinning object, we add up the (orbital) kinetic energies of the various parts. The result is:

$$KE_{total} = \frac{1}{2} I\omega^2$$

Rotational Distortion

- Although we think of Earth as a solid object, it is able to "flow" in response to stresses and it has a shape that is approximately the same as that of a spinning, self-gravitating fluid: an **oblate ellipsoid**. Newton recognized this, but confirming measurements were first made after Newton's death.

Reese Ch. 10, part A3
(pp. 445-455)

- Precession
- Simultaneous Spin and Orbital Motion
- Synchronous Rotation &
  Parallel-Axis Theorem
- Pure Rolling Motion

Precession

- If a rapidly spinning top is subjected to a torque that is perpendicular to its spin axis, the top slowly precesses: the direction of its symmetry axis gradually moves in the direction of the torque.
- For a given torque, the precession rate is inversely proportional to the spin rate: the faster a top spins, the more slowly it precesses.

Earth's Precession

- The Earth precesses because the Moon and Sun exert a net external torque (due to the Earth's equatorial bulge and the fact that Earth's equator does not lie in the plane of its orbit).
- One precessional cycle takes nearly 26,000 years: the angular speed of Earth's precession is only about 100 parts per billion of its spin angular speed!

Simultaneous Spin and Orbital Motion: I

The total KE of a system can be written as the sum of the spin KE and the translational KE associated with the motion of the system's center of mass:

$$KE_{total} = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v_{CM}^2$$

Simultaneous Spin and Orbital Motion: II

The total angular momentum of a system can be written as the vector sum of the spin angular momentum and the angular momentum associated with the motion of the CM of the system:

$$\vec{L}_{total} = I_{CM} \vec{\omega}_{spin} + M\vec{r}_{CM} \times \vec{v}_{CM}$$

Simultaneous Spin and Orbital Motion: III

The angular momentum associated with the motion of the CM of the system can also be written as:

$$\vec{L}_{orbital} \equiv M\vec{r}_{CM} \times \vec{v}_{CM} = Mr_{\perp CM}^2 \vec{\omega}_{orbital}$$

Then the total angular momentum is:

$$\vec{L}_{total} = I_{CM} \vec{\omega}_{spin} + Mr_{\perp CM}^2 \vec{\omega}_{orbital}$$

Synchronous Rotation I

- If the spin angular velocity and orbital angular velocity have the same magnitude and direction $\vec{\omega}_{spin} = \vec{\omega}_{orbital} = \vec{\omega}$

  then we say that the object exhibits synchronous rotation.
- An example is the Moon, which exhibits (nearly) synchronous rotation as it orbits the Earth.

Synchronous Rotation II

- In the case of synchronous rotation, the body's total angular momentum simplifies to

$$\vec{L}_{total} = (I_{CM} + Mr_{\perp CM}^2)\vec{\omega}$$

- In synchronous rotation, rather than rotating about its symmetry axis, the body is effectively rotating about a parallel axis through the coordinate origin. Referring to the equation above, the rotational inertia about this "displaced axis" is $I = I_{CM} + Mr_{\perp CM}^2$
This is called the parallel-axis theorem.

Pure Rolling Motion

- Consider a rigid wheel (or other symmetrical rigid body of circular cross section). When it rolls along a surface without slipping, the point of its contact with the surface is the (instantaneous) axis of a synchronous rotation:

Pure rolling is one type of synchronous rotation!

Equations That Describe Pure Rolling on a Flat Surface

- When a rolling wheel of radius R rotates through an angle θ, the hub (center) of the wheel moves a distance s = R θ.
- The hub's speed $v_{center} = R\frac{d\theta}{dt} = R\omega$
- The linear acceleration of the hub is given by

$$a_{center} = R\frac{d^2\theta}{dt^2} = R\frac{d\omega}{dt} = R\alpha$$

Total Kinetic Energy of a Rolling Wheel

- We apply our earlier result that the total KE is the sum of spin KE and the KE associated with translation of the CM:

$$KE_{total} = KE_{spin} + KE_{translation}$$

$$\Rightarrow KE_{total} = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}Mv_{CM}^2$$

$$\Rightarrow KE_{total} = \frac{1}{2}(I_{CM} + MR^2)\omega^2 = \frac{1}{2}I\omega^2$$

(The last step uses the parallel-axis theorem.)