INTRODUCTION TO THE

VECTOR DIFFERENTIAL OPERATOR "DEL"

1. The Directional Derivative

Consider a function f which depends upon three independent variables (x,y,z). For the purposes of visualization, you can imagine these to be the three Cartesian coordinates that specify a position in space. If f(x,y,z) is a sufficiently smooth function, then it possesses at each point $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and for each direction (as specified by a unit vector $\hat{n}$ pointing in the desired direction) a **directional derivative.** The directional derivative is the rate of change of $f(\vec{r})$ with distance in the direction specified by $\hat{n}$. The directional derivative is denoted by $\frac{df}{ds}(\vec{r};\hat{n})$ and is given by

$$\frac{df}{ds}(\vec{r};\hat{n}) \equiv \lim_{\Delta s \to 0}\left[\frac{f(\vec{r}+\hat{n}\Delta s) - f(\vec{r})}{\Delta s}\right] \qquad \text{(Eq. 1)}$$

We need to determine the relationship between $\frac{df}{ds}(\vec{r};\hat{n})$ and the **partial derivatives** of $f(\vec{r})$ with respect to each of the independent variables. These partial derivatives are <u>defined</u> by

$$\frac{\partial f}{\partial x}(\vec{r}) \equiv \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}\right] \qquad \text{(Eq. 2a)}$$

$$\frac{\partial f}{\partial y}(\vec{r}) \equiv \lim_{\Delta y \to 0}\left[\frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y}\right] \qquad \text{(Eq. 2b)}$$

$$\frac{\partial f}{\partial z}(\vec{r}) \equiv \lim_{\Delta z \to 0}\left[\frac{f(x, y, z+\Delta z) - f(x,y,z)}{\Delta z}\right] \qquad \text{(Eq. 2c)}$$

To establish the relationship between the directional

whi
9/82

derivative and these partial derivatives, we write out the quantity whose limit is $\frac{df}{ds}(\vec{r};\hat{n})$. Because the unit vector $\hat{n}$ can be written as $\hat{n} = \hat{x}(\hat{n}\cdot\hat{x}) + \hat{y}(\hat{n}\cdot\hat{y}) + \hat{z}(\hat{n}\cdot\hat{z})$, we have

$$f(\vec{r}+\hat{n}\Delta s) = f\left[x+(\hat{n}\cdot\hat{x})\Delta s,\ y+(\hat{n}\cdot\hat{y})\Delta s,\ z+(\hat{n}\cdot\hat{z})\Delta s\right]$$

Therefore the quantity whose limit is the directional derivative has the form

$$\frac{1}{\Delta s}\left[f(\vec{r}+\hat{n}\Delta s)-f(\vec{r})\right] = \frac{1}{\Delta s}\left\{f\left[x+(\hat{n}\cdot\hat{x})\Delta s, y+(\hat{n}\cdot\hat{y})\Delta s, z+(\hat{n}\cdot\hat{z})\Delta s\right] - f\left[x,y,z\right]\right\}$$

By adding and subtracting equal terms and regrouping, we can rewrite this as $\frac{1}{\Delta s}\left[f(\vec{r}+\hat{n}\Delta s)-f(\vec{r})\right]$

$$= \frac{1}{\Delta s}\left\{\begin{array}{l} f\left[x+(\hat{n}\cdot\hat{x})\Delta s, y+(\hat{n}\cdot\hat{y})\Delta s, z+(\hat{n}\cdot\hat{z})\Delta s\right] \\ \quad - f\left[x, y+(\hat{n}\cdot\hat{y})\Delta s, z+(\hat{n}\cdot\hat{z})\Delta s\right]\end{array}\right\} + \frac{1}{\Delta s}\left\{\begin{array}{l} f\left[x, y+(\hat{n}\cdot\hat{y})\Delta s, z+(\hat{n}\cdot\hat{z})\Delta s\right] \\ \quad - f\left[x, y, z+(\hat{n}\cdot\hat{z})\Delta s\right]\end{array}\right\}$$
$$+ \frac{1}{\Delta s}\left\{f\left[x,y,z+(\hat{n}\cdot\hat{z})\Delta s\right] - f\left[x,y,z\right]\right\}$$

If we look carefully at each term in the above equation, we can see that in the limit $\Delta s \to 0$, the equation becomes

$$\frac{df}{ds}(\vec{r};\hat{n}) = (\hat{n}\cdot\hat{x})\frac{\partial f(\vec{r})}{\partial x} + (\hat{n}\cdot\hat{y})\frac{\partial f(\vec{r})}{\partial y} + (\hat{n}\cdot\hat{z})\frac{\partial f(\vec{r})}{\partial z} \qquad \text{(Eq. 3)}$$

This is the desired expression for $\frac{df}{ds}(\vec{r};\hat{n})$ in terms of the partial derivatives of $f(\vec{r})$.

2. The Gradient of a Scalar Function

Equation 3 has the form of a **dot product** between the vector $\hat{n}$ and the following vector:

$$\hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

This vector is called the **gradient** of the scalar function $f(\vec{r})$ and is denoted by **grad f** or by $\vec{\nabla}$**f**. Thus the directional derivative of f can be written

$$\frac{df}{ds}(\vec{r};\hat{n}) = \hat{n}\cdot \operatorname{grad} f(\vec{r}) = \hat{n}\cdot\vec{\nabla}f(\vec{r}) \qquad \text{(Eq. 4)}$$

By carefully considering eq. 4, you should be able to convince yourself that the gradient of $f(\vec{r})$ points in the direction of most rapid increase of f with distance. You should commit eq. 4 to memory, and you should also memorize the definition of the gradient of $f(\vec{r})$:

$$\vec{\nabla}f(\vec{r}) \equiv \hat{x}\frac{\partial f(\vec{r})}{\partial x} + \hat{y}\frac{\partial f(\vec{r})}{\partial y} + \hat{z}\frac{\partial f(\vec{r})}{\partial z} \qquad \text{(Eq. 5)}$$

Keep firmly in mind that the gradient operation produces a vector function from a scalar function.

3. Del, Div, and Curl

The gradient of a scalar functions is one of several useful derivatives that can easily be written in terms of the vector operator **del**, which is defined by

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \qquad \text{(Eq. 6)}$$

We now mention two other derivatives involving del, namely **div** and **curl**.

The **divergence** of a vector function $\vec{u}(\vec{r})$ is given by

$$\text{div}\ \vec{u}(\vec{r}) \equiv \vec{\nabla}\cdot\vec{u}(\vec{r}) \equiv \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(u_x\hat{x} + u_y\hat{y} + u_z\hat{z}\right)$$

$$\equiv \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \qquad \text{(Eq. 7)}$$

Notice that the divergence operation produces a scalar function from a vector function. The divergence of a vector function is a measure of the local tendency of the vector function (or "field of vectors") to spread out or diverge.

The **curl** or **rotation** of a vector function u(r) is given by

$$\text{curl}\ \vec{u}(\vec{r}) \equiv \vec{\nabla}\times\vec{u}(\vec{r}) \equiv \hat{x}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) + \hat{y}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + \cdots$$

$$\cdots + \hat{z}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \equiv \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_x & u_y & u_z \end{vmatrix} \qquad \text{(Eq. 8)}$$

The curl operation produces a vector function from a vector function. The curl of a vector function is a measure of the local tendency of the field of vectors to coil around.

4. Three Important Theorems Involving Del

In the calculus of functions of one variable, a central result is the **fundamental theorem of calculus:**

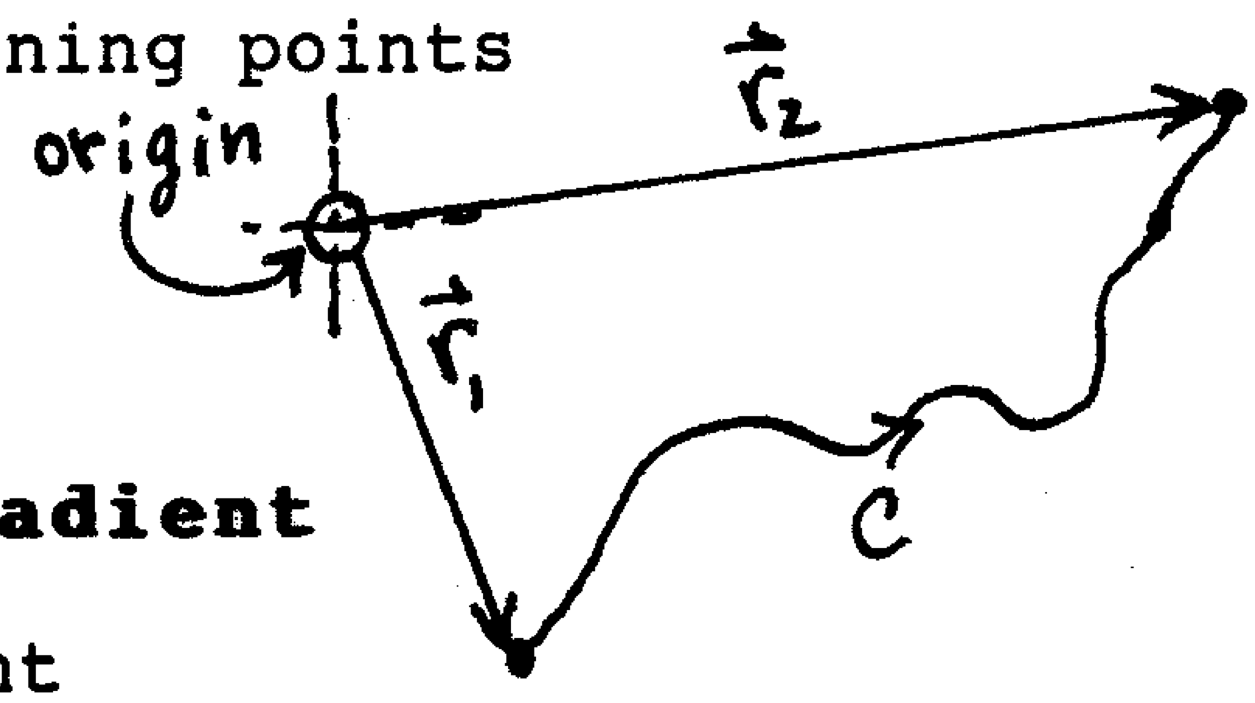
> The indefinite integral of the derivative of a function is equal to the function itself, to within an additive constant.

In symbolic form, the fundamental theorem is:

$$\int^{x} \frac{df(x')}{dx'}\,dx' = f(x) + C$$

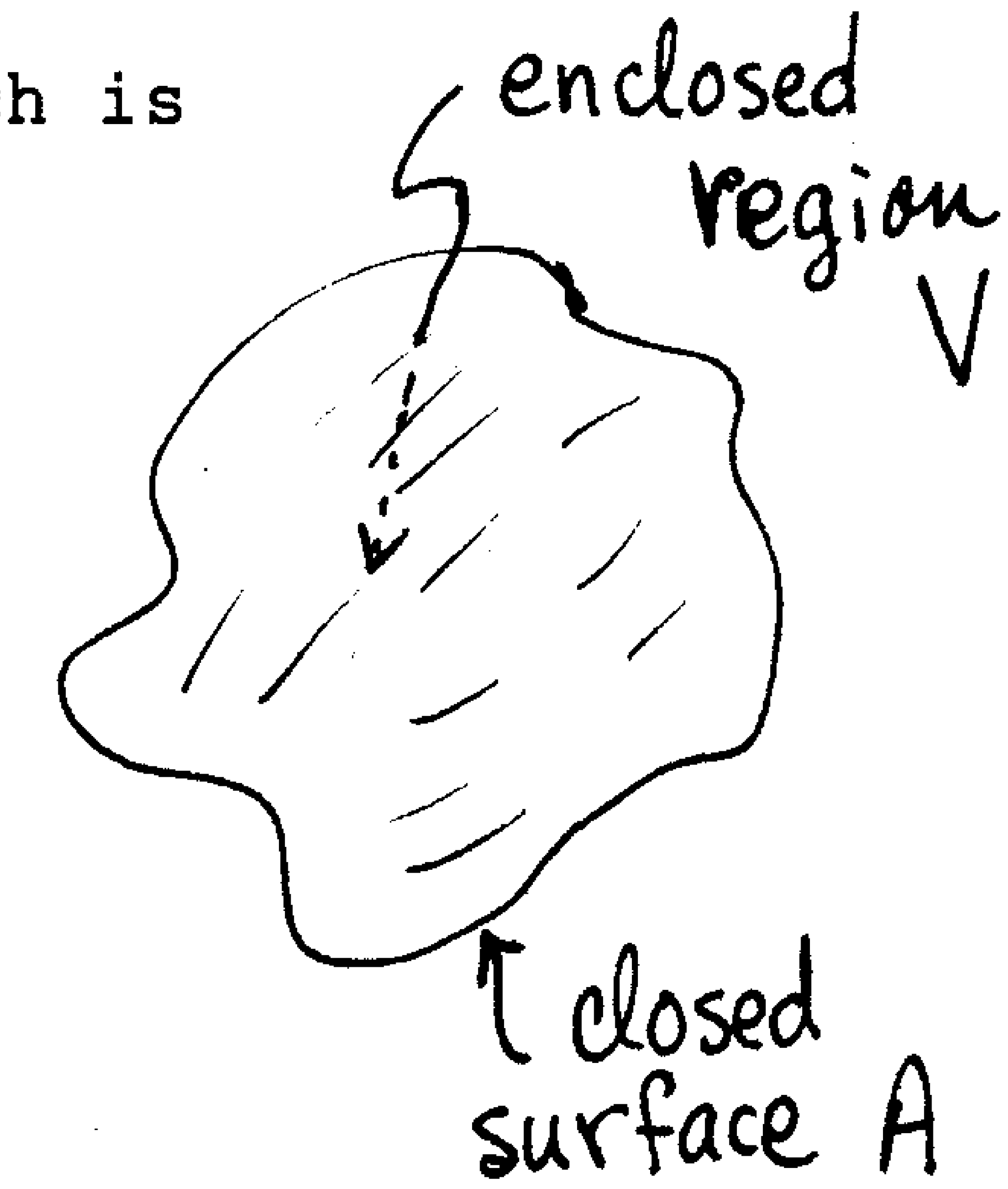
In the calculus of functions of three variables, there are several results similar to this one, in which a function is related to the integral of a derivative. **These theorems play a very important role in the statement of various principles of physics.** We present three of these theorems here (without proof): the gradient theorem, the divergence theorem, and Stokes' theorem (or the curl theorem).

__The Gradient Theorem.__ Suppose that we are given a scalar function $f(\vec{r})$ and __any__ curve C joining points $\vec{r}_1$ and $\vec{r}_2$, as shown. Then **the difference between $f(\vec{r}_2)$ and $f(\vec{r}_1)$ equals the path integral of the gradient of $f(\vec{r})$.** In symbolic form, the gradient theorem reads



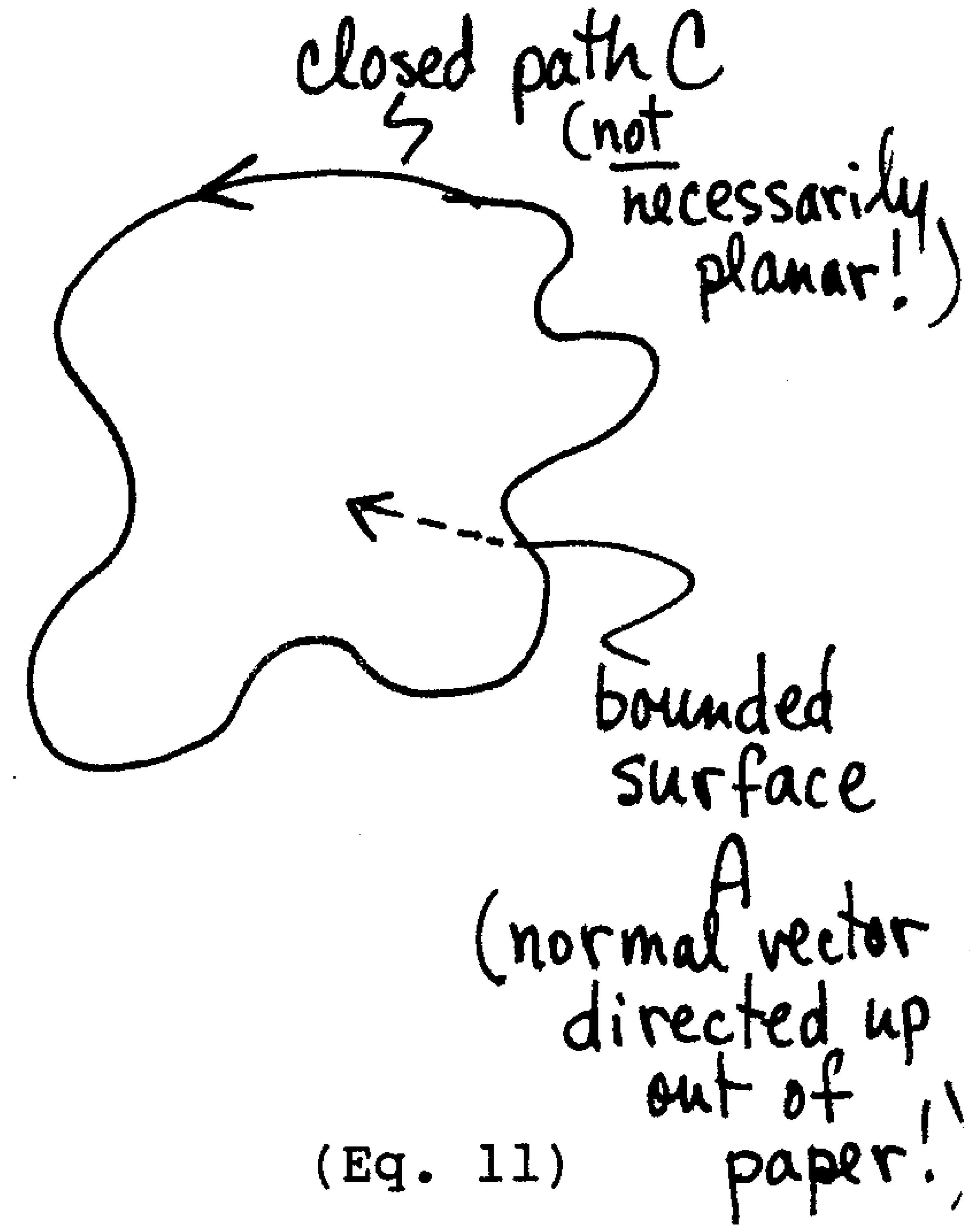
$$f(\vec{r}_2) - f(\vec{r}_1) = \int_{C\ \vec{r}_1}^{\vec{r}_2} \left[\vec{\nabla} f(\vec{r})\right]\cdot d\vec{r} \qquad \text{(Eq. 9)}$$

The Divergence Theorem. Suppose that we are given a vector function $\vec{u}(\vec{r})$ and a closed surface A which is the boundary of a region V of space, as shown. Then **the flux of the vector function $\vec{u}(\vec{r})$ through the closed surface A equals the volume integral of the divergence of $\vec{u}(\vec{r})$ over the region V.** In symbolic form, the divergence theorem reads



$$\underset{\substack{\text{closed surface}\\ A}}{\oint\!\!\oint} \vec{u}(\vec{r})\cdot d\vec{A} = \underset{\substack{\text{enclosed}\\ \text{volume } V}}{\iiint} \left[\vec{\nabla}\cdot\vec{u}(\vec{r})\right] dV \qquad \text{(Eq. 10)}$$

Stokes' Theorem, or The Curl Theorem. Suppose that we are given a vector function $\vec{u}(\vec{r})$ and a closed path C, which is the boundary of an (open) surface A, as shown. Then **the counterclockwise line integral of $\vec{u}(\vec{r})$ around the path C equals the surface integral of the curl of $\vec{u}(\vec{r})$ over the surface A.** In symbolic form, Stokes' theorem reads



$$\underset{\substack{\text{closed curve}\\ C}}{\oint} \vec{u}(\vec{r})\cdot d\vec{r} = \underset{\substack{\text{bounded}\\ \text{surface } A}}{\iint} \left[\vec{\nabla}\times\vec{u}(\vec{r})\right]\cdot d\vec{A} \qquad \text{(Eq. 11)}$$

5. Second-order Derivatives Involving Del

Often more than one operation involving del is applied to a function. One important example is **the divergence of the gradient of a scalar function f($\vec{r}$)**: this is called **div(grad f)** or $\vec{\nabla}\cdot(\vec{\nabla} f)$. Using the definitions, it is straightforward to show that div(grad f) equals

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

(You should check this claim!) This particular combination of second partial derivatives of f(r) is called the **Laplacian** of f and is usually called **del-squared f** and written $\nabla^2 f$:

$$\nabla^2 f \equiv \vec{\nabla}\cdot(\vec{\nabla} f) \equiv \mathrm{div}(\mathrm{grad}\, f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{(Eq. 12)}$$

There are two second-order derivatives involving del which are identically zero. These vanishing derivatives are the **curl of the gradient** of any scalar function f($\vec{r}$) and the **divergence of the curl** of any vector function $\vec{u}(\vec{r})$:

$$\mathrm{curl}\,(\mathrm{grad}\, f) \equiv \vec{\nabla}\times(\vec{\nabla} f) \equiv \vec{0} \qquad \text{(Eq. 13)}$$

$$\mathrm{div}\,(\mathrm{curl}\, \vec{u}) \equiv \vec{\nabla}\cdot(\vec{\nabla}\times\vec{u}) = 0 \qquad \text{(Eq. 14)}$$

(You should prove these identities by applying the definitions.)